Exercise 2

Use power series to solve the differential equation.

$$y' = xy$$

Solution

$x = 0$ is an ordinary point, so the ODE has a power series solution centered here.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate the series with respect to x.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substitute these formulas into the ODE.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = x \sum_{n=0}^{\infty} a_n x^n$$

Bring x inside the summand.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^{n+1}$$

Make the substitution $n = k + 2$ in the series on the left and the substitution $n = k$ in the series on the right.

$$\sum_{k+2=1}^{\infty} (k+2) a_{k+2} x^{(k+2)-1} = \sum_{k=0}^{\infty} a_k x^{k+1}$$

Simplify the sum on the left.

$$\sum_{k=-1}^{\infty} (k+2) a_{k+2} x^{k+1} = \sum_{k=0}^{\infty} a_k x^{k+1}$$

Write out the first term on the left.

$$(-1+2) a_{-1+2} x^{-1+1} + \sum_{k=0}^{\infty} (k+2) a_{k+2} x^{k+1} = \sum_{k=0}^{\infty} a_k x^{k+1}$$

Combine the series on the left side.

$$a_1 + \sum_{k=0}^{\infty} \left[(k+2) a_{k+2} - a_k\right] x^{k+1} = 0$$

In order for the left side to be zero, both a_1 and the quantity in square brackets must be zero.

$$(k+2) a_{k+2} - a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{1}{k+2}a_k \qquad a_1 = 0$$

In order to determine a_k, plug in values for k and try to find a pattern.

$$k = 0: \quad a_2 = \frac{1}{0+2}a_0 = \frac{1}{2}a_0$$

$$k = 1: \quad a_3 = \frac{1}{1+2}a_1 = 0$$

$$k = 2: \quad a_4 = \frac{1}{2+2}a_2 = \frac{1}{4}\left(\frac{1}{2}a_0\right) = \frac{1 \cdot 1}{4 \cdot 2}a_0$$

$$k = 3: \quad a_5 = \frac{1}{3+2}a_3 = 0$$

$$k = 2: \quad a_6 = \frac{1}{4+2}a_4 = \frac{1}{6}\left(\frac{1 \cdot 1}{4 \cdot 2}a_0\right) = \frac{1 \cdot 1 \cdot 1}{6 \cdot 4 \cdot 2}a_0$$

$$\vdots$$

The general formula for the even subscripts is

$$a_{2m} = \frac{1}{(2m)!!}a_0 = \frac{1}{2^m m!}a_0,$$

and the general formula for the odd subscripts is

$$a_{2m+1} = 0.$$

Therefore, the general solution is

$$\begin{aligned} y(x) &= \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} a_{2m}x^{2m} + \sum_{m=0}^{\infty} a_{2m+1}x^{2m+1} \\ &= \sum_{m=0}^{\infty} \frac{1}{2^m m!}a_0 x^{2m} + \sum_{m=0}^{\infty} (0)x^{2m+1} \\ &= a_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{2^m m!} \\ &= a_0 \sum_{m=0}^{\infty} \frac{\left(\frac{x^2}{2}\right)^m}{m!} \\ &= a_0 e^{x^2/2}, \end{aligned}$$

where a_0 is an arbitrary constant.